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SOME ESTIMATES FOR A DECOMPOSITION OF HYPERSURFACE SINGULARITIES INTO THE SIMPLEST ONES

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ABSTRACT
The upper and lower bounds are obtained for the maximal number of Morse points with the same critical value into which an isolated hypersurface singularity can be decomposed.

1. Introduction

Let $\delta(f)$ be the maximal possible number of Morse points with the same critical value, into which an isolated singularity of an analytic germ $f:(C^m,0)\to(C,0)$ can be decomposed. Let us define also $a(f)$ as $\max\{k \mid \text{the singularity of } f \text{ abuts a singularity of type } A_k\}$.

In this paper some inequalities are obtained for numbers $\delta(f)$ and $a(f)$. The lower bounds for $a(f)$ and then for $\delta(f)$ are obtained by the explicit construction of deformation of the singularity, which gives the required decomposition. The consideration of the intersection matrix of the singularity gives the upper bounds for δ. As a consequence a weaker inequality is obtained, which contains only Milnor numbers of the singularity and of its generic hyperplane section.

2. Some definitions

Let $f:(C^m,0)\to(C,0)$ be the germ of an anlytic function with an isolated singularity at the origin. The Milnor number $\mu(f)$ of this singularity is defined as

$$\mu(f)=\dim_C O_{m,0}\Big/\left\{\frac{\partial f}{\partial z_1},\cdots,\frac{\partial f}{\partial z_m}\right\}, \tag{1}$$

where $O_{m,0}$ is the ring of germs of analytic functions at $0\in C^m$, and $\{\partial f/\partial z_1,\cdots,\partial f/\partial z_m\}$ is the ideal, generated by all partial derivatives of f.

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Two singularities $f:(C^m,0)\to(C,0)$ and $g:(C^m,0)\to(C,0)$ are said to be equivalent (or of the same type) if there exists a germ of an analytic diffeomorphism $\phi:(C^m,0)\to(C^m,0)$ such that $g=f\circ\phi$.

Singularities $f:(C^m,0)\to(C,0)$ and $g:(C^k,0)\to(C,0)$, $m\geqq k$, are called stably equivalent if there exists a local coordinate system $w_1,\cdots,w_m$ at $0\in C^m$ such that

$$f(w_1,\cdots,w_m)=f_1(w_1,\cdots,w_k)+w_{k+1}^2+\cdots+w_m^2,$$

and f_1 is equivalent to g.

The following well-known fact will be used:

PROPOSITION 1. *For $f:(C^m,0)\to(C,0)$ let $\operatorname{rank}(\partial^2 f/\partial z_i\partial z_j)_0=q$, $q<m$. Then the singularity of f is stably equivalent to the singularity of the function f_1 of $m-q$ variables, and the Hessian of f_1 at 0 is equal to zero.*

The singularity with the Milnor number μ, which is stably equivalent to the singularity of the function of one variable, is said to be of type A_μ. By a coordinate change it can be reduced to the form

$$f=z_1^{\mu+1}+z_2^2+\cdots+z_m^2.$$

The singularity of type A_1 is called a Morse point.

Let $f:(C^m,0)\to(C,0)$ be as above. The germ of the analytic function $F:(C^m\times C,0)\to(C,0)$, for which $F(z,0)\equiv f(z)$ is called the deformation of f with base C.

The singularity of f under the deformation F can be decomposed with respect to the type $X=(X_1,\cdots,X_k)$, where $X_i=(X_{i,1},\cdots,X_{i,l_i})$, if for all $t\in C$, $0<|t|\ll 1$, the function $f_t=F(\cdot,t)$ in small neighbourhood of $0\in C^m$ has exactly k different critical values, the ith critical value being attained at critical points of types $X_{i,1},\cdots,X_{i,l_i}$. In this case $\mu(X)=\Sigma_{i,j}\mu(X_{i,j})$.

A singularity of type X abuts a collection of singularities $Y=(Y_1,\cdots,Y_l)$ if X can be decomposed with respect to the type $(Y,\cdots)$.

3. Definition of δ and a

DEFINITION. For an isolated singularity of type X let $\delta(X)$ be the maximal k for which X abuts a collection $Y=(\underbrace{A_1,\cdots,A_1}_{k})$. Let $a(X)$ be the maximal l, for which X abuts $Y=\{A_l\}$.

The following properties of δ and a can be easily proved:

(i) If singularities X and Y are stably equivalent, then $\delta(X)=\delta(Y)$, $a(X)=A(Y)$.

(ii) If X abuts Y, then $\delta(X) \geqq \delta(Y)$, $a(X) \geqq a(Y)$.

(iii) $\delta(A_l) = [(l+2)/2]$.

From (ii) and (iii) it follows that

(iv) $\delta(X) \geqq [(a(X)+1)/1]$.

For the case $m = 2$, i.e. for singularities of plane curves, δ is a number of "double point" of the curve $f = 0$ at 0, and is given by the formula

$$\delta = \tfrac{1}{2}(\mu + r - 1),$$

where μ is the Milnor number and r the number of branches of the curve $f = 0$ at 0 ([2]). In this case also $\delta = \dim_c \bar{O}/O$, where O is a local ring of the curve $f = 0$ at 0, and $\bar{O}$ its normalization ([5]).

4. The lower bound for a and δ

Let $f:(C^m, 0) \to (C, 0)$ be as above and let CP^{m-1} be the space of hyperplanes in C^m passing through the origin. It is proved in [3] that there exists a Zariski-open dense subset $U \subset CP^{m-1}$ such that for $L \in U$, the restriction f/L has an isolated singularity at the origin. The Milnor number of this singularity is independent of $L \in U$ and is denoted by $\mu^{(m-1)}(f)$.

THEOREM 1. $a(f) \geqq \mu(f)/\mu^{(m-1)}(f)$.

PROOF. The set $U \subset CP^{m-1}$ above can be chosen in such a way that the following is true:

Let us fix $L \in U$ and choose coordinates $z_1, \cdots, z_m$ so that L is given by $z_1 = 0$. Then the curve Γ_L defined by equations

$$\frac{\partial f}{\partial z_2} = 0, \cdots, \frac{\partial f}{\partial z_m} = 0 \qquad \text{(the polar curve)}$$

is reduced and has a number of irreducible components l in its decomposition $\Gamma_L = \bigcup \Gamma_q$ which is independent of $L \in U$. Furthermore, setting $m_q = (\Gamma_q, L)_0$, $e_q = (\Gamma_q, \{\partial f/\partial z_1 = 0\})_0$ (where $(\ \ ,\ \ )_0$ is the intersection multiplicity), we have a sequence of integers e_q, m_q, which is also independent of $L \in U$ and

$$\sum e_q = \mu(f), \qquad \sum m_q = \mu^{(m-1)}(f) \qquad \text{(see [4])}.$$

Let $\alpha = [\sup_q (e_q/m_q)]$. It follows that $\alpha \geqq \mu/\mu^{(m-1)}$. Take a component Γ_q, on which this supremum is attained, and the pont $x \in \Gamma_q \setminus \{0\}$, and consider z_1 as a local coordinate on Γ_q in a neighbourhood of x. Let $P_x(z_1)$ be the Teilor polynomial of f/Γ_q at x of the degree α. Then $f - P_x(z_1)/\Gamma_q$ vanishes with multiplicity $k \geqq \alpha + 1$ at x.

LEMMA 1. *The function $f_x = f - P_x(z_1)$ has a singularity of type A_{k-1} at x.*

PROOF. Since P is a polynomial in one variable z_1,

$$\frac{\partial f_x}{\partial z_i} = \frac{\partial f}{\partial z_i}, \qquad i = 2, \cdots, m,$$

hence

$$\operatorname{rank}\left(\frac{\partial^2 f_x}{\partial z_i \partial z_j}\right)_{\substack{i \geqq 1 \\ j \geqq 1}} \geqq \operatorname{rank}\left(\frac{\partial^2 f_x}{\partial z_i \partial z_j}\right)_{\substack{i \geqq 2 \\ j \geqq 1}} = \operatorname{rank}\left(\frac{\partial^2 f}{\partial z_i \partial z_j}\right)_{\substack{i \geqq 2 \\ j \geqq 1}} = m - 1,$$

where the last equality follows from the fact that the curve Γ_q, defined in the neighbourhood of x by equations $\partial f / \partial z_2 = 0, \cdots, \partial f / \partial z_m = 0$, is reduced. Now by Proposition 1 the singularity of f_x at x is of type A_μ, where μ is its Milnor number. But in this case the number μ can be easily computed by formula (1) — it is equal to $k - 1$.

Now it only remains to prove that all the coefficients of P_x tend to zero as $x \to 0$ on Γ_q. Let $n : D \to \Gamma_q$ be the normalization of Γ_q (D a neighbourhood of $0 \in C$). Then from $(\Gamma_q, L)_0 = m_q$, $(\Gamma_q, \{\partial f / \partial z_1 = 0\})_0 = e_q$ it follows that we can choose the coordinate t on D such that

$$z_1 \circ n = t^{m_q},$$

$$\frac{\partial f}{\partial z_1} \circ n = \beta t^{e_q} + \cdots, \qquad \beta \neq 0.$$

But

$$\frac{d}{dt}(f \circ n) = \sum_{i=1}^{m} \frac{\partial f}{\partial z_i} \cdot \frac{dz_i}{dt} = \frac{\partial f}{\partial z_1} \cdot \frac{dz_1}{dt} \quad \left(\frac{\partial f}{\partial z_2} = \cdots = \frac{\partial f}{\partial z_m} = 0 \quad \text{on } \Gamma_q\right),$$

hence

$$f \circ n = \frac{\beta m_q}{m_q + e_q} t^{m_q + e_q} + \cdots$$

and as $t = z_1^{1/m_q}$, we have $f/\Gamma_\nu = \gamma z_1^{1+e_q/m_q} + \cdots$, $\gamma \neq 0$.

Hence all the derivatives of f/Γ_ν with respect to z_1 up to α-th tend to zero as $z_1 \to 0$, and hence the coefficients of the Teilor polynomial P_x tend to zero as $x \to 0$.

COROLLARY 1. $2\delta(f) \geqq a(f) \geqq \mu(f)/\mu^{(m-1)}(f)$.

5. The upper bound for δ

To formulate the following theorem we need some results on the local topology of isolated singularities ([2]).

Let $f:(C^m,0)\to(C,0)$ be as above and let D_ε and S_ε be an open disk and sphere of radius ε centered at $0\in C^m$. Then if $0<\varepsilon\ll 1$, $K=f^{-1}(0)\cap S_\varepsilon$ is a smooth $m-3$-connected manifold (of dimension $2m-3$), and the mapping $f/|f|:S_\varepsilon\backslash K\to S^1$ is a smooth fibering (the Milnor fibering). Its fiber F is a smooth $2m-2$-dimensional manifold, diffeomorphic to the manifold $D_\varepsilon\cap f^{-1}(\xi)$, $0<|\xi|\ll\varepsilon$, and homotopy equivalent to the wedge of $\mu(f)$ spheres S^{m-1} Let $h:F\to F$ be the characteristic mapping of this fibering.

Similarly if there is one more function $f_1:(C^m,0)\to(C,0)$, such that the variety $Y=f^{-1}(0)\cap f_1^{-1}(0)$ has an isolated singularity at $0\in C^m$, $K_1=Y\cap S_\varepsilon$ is now $m-4$-connected manifold ($\dim K_1=2m-5$), and $f_1/|f_1|:K\backslash K_1\to S^1$ is a smooth fibering. Its fiber F_1 is diffeomorphic to the manifold $D_\varepsilon\cap f^{-1}(\xi)\cap f_1^{-1}(\xi_1)$, where $(\xi,\xi_1)\in C^2$ is sufficiently small regular value of $(f,f_1):(C^m,0)\to(C^2,0)$. F_1 is also homotopy equivalent to the wedge of spheres S^{m-2}.

We shall consider the reduced homology groups with complex coefficients.

For given singularity $f:(C^m,0)\to(C,0)$ let us consider the intersection form ω on $H_{m-1}(F)$. Let $p(f)$ be corank ω (equal to $\dim\operatorname{Ker}(h_*-I)$: $H_{m-1}(F)\to H_{m-1}(F)$ or to $\dim H_{m-1}(K)$) ([2]).

THEOREM 2. $2\delta(f)\leqq\mu(f)+p(f)$.

PROOF. Replacing the singularity f by the stably equivalent one, we can suppose that m is even and hence the form ω is skew-symmetric. By definition of δ, the singularity of f can be decomposed into μ Morse points, from which $\delta(f)$ have the same critical value. The vanishing cycles, corresponding to these $\delta(f)$ Morse points ([1]) are pairwise nonintersecting. Hence the restriction of the form ω on the $\delta(f)$-dimensional subspace $M\subset H_{m-1}(F)$, generated by these cycles, is zero (self-intersections of cycles are zero since ω is skew-symmetric). Let N be the zero subspace of ω, then $\dim N=p$ and ω defines a nondegenerated form $\tilde{\omega}$ on $R=H_{m-1}(F)/N$, and $\tilde{\omega}/\tilde{M}\equiv 0$, where $\tilde{M}$ is the image of M in R.

Let $s=\dim\tilde{M}$, $s\geqq\delta(f)-p$. For the basis $e_1,\cdots,e_s$ of $\tilde{M}$ let $e'_1,\cdots,e'_s$ be such that $\tilde{\omega}(e_i,e'_j)=\delta_{ij}$, and let M' be the subspace, generated by $e'_1,\cdots,e'_s$. Then $M'\cap\tilde{M}=0$, since $\tilde{\omega}/\tilde{M}=0$. Hence $\dim R=\mu-p\geqq 2s\geqq 2(\delta(f)-p)$, and $2\delta(f)\leqq\mu+p$.

THEOREM 3. $p(f)\leqq\mu^{(m-1)}(f)$.

PROOF. Let $L = \{l = 0\}$ be the generic hyperplane in C^m, passing through the origin. Let $K = S_\varepsilon \cap f^{-1}(0)$, $K_1 = K \cap L = S_\varepsilon \cap f^{-1}(0) \cap l^{-1}(0)$, and let h_1 be the characteristic mapping of the fibering $l/|l| \cdot K \backslash K_1 \to S^1$, F_1 its fiber.

The exact sequence

$$\to H_{m-1}(K_1) \to H_{m-1}(K) \to H_{m-1}(K, K_1) \to H_{m-2}(K_1) \to$$

where $H_{m-1}(K_1) = H^{m-4}(K_1) = 0$, gives now:

$$p = \dim H_{m-1}(K) \leqq \dim H_{m-1}(K, K_1) = \dim H^{m-2}(K \backslash K_1) = \dim H_{m-2}(K \backslash K_1).$$

The last group can be included in the Wang exact sequence of the fibering

$$l/|l| : K \backslash K_1 \to S^1 : H_{m-2}(F_1) \xrightarrow{h_{1*}-I} H_{m-2}(F_1) \to H_{m-2}(K \backslash K_1) \to H_{m-3}(F_1) \to .$$

The fiber F_1 can be identified with the manifold $D_\varepsilon \cap L \cap f^{-1}(\xi)$, $0 < |\xi| \ll 1$, i.e. with the Milnor fiber of singularity of f/L. Then $H_{m-3}(F_1) = 0$, $\dim H_{m-2}(F_1) = \mu^{(m-1)}(f)$, and we have:

$$p \leqq \dim H_{m-2}(K \backslash K_1) \leqq \dim H_{m-2}(F_1) = \mu^{(m-1)}.$$

COROLLARY 2. $2\delta(f) \leqq \mu(f) + \mu^{(m-1)}(f)$.

REMARK 1. Another proof of the last inequality was given by B. Teissier [4].

REMARK 2. For the case $m = 2$ we have an equality $\delta(f) = (\mu + p)/2$, since p in this case is equal to $r - 1$, where r is the number of branches of the curve $f = 0$ at 0. $\mu^{(1)}$ in this case can be strictly less than p.

6. Some consequences

We have now the following estimates for δ and a:

$$(*) \qquad \delta \geqq \left[\frac{a+1}{2}\right], \quad a \geqq \frac{\mu}{\mu^{(m-1)}}, \qquad 2\delta \leqq \mu + p,$$

and since $\mu^{(m-1)} \leqq p$, $\mu/\mu^{(m-1)} \leqq 2\delta \leqq \mu + \mu^{(m-1)}$.

We can also derive an estimate for δ and a in terms of μ only:

COROLLARY 3. *Let* $q = m - \operatorname{rank}(\partial^2 f/\partial z_i \partial z_j)$. *Then* $\mu^{1/q} \leqq 2\delta \leqq \mu(1 + 1/\mu^{1/q})$, $a \geqq \mu^{1/q}$.

PROOF. From Proposition 1 the singularity of f at 0 is stably equivalent to the singularity of function f_1 of q variables, and $\delta(f) = \delta(f_1)$, $a(f) = a(f_1)$. For f_1 we have inequalities (*), with $\mu^{(q-1)}$ instead $\mu^{(m-1)}$. But by [3],

$$\frac{\mu}{\mu^{(q-1)}} \geqq \mu^{1/q}.$$

Corollary 4. *Let m_0 be the multiplicity of f_1 at 0. Then*

$$m_0 - 1 \leqq 2\delta(f) \leqq \mu\left(1 + \frac{1}{m_0 - 1}\right).$$

Proof. Also by [3],

$$\frac{\mu}{\mu^{(q-1)}} \geqq m_0 - 1.$$

It follows from Proposition 1 that $m_0(f_1) \geqq 3$, if $q \neq 0$. Then we have

Corollary 5. *For all singularities except A_1, $\delta \leqq \frac{3}{4}\mu$.*

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